Pontryagin Maximum Principle

① Geometric PMP

② Optimal control PMP

① Geometric PMP

Control problem

$$\dot{q} = V_u(q) \quad , \quad q \in M \quad , \ u \in U$$
$$q(0) = q_0$$

where M mfd., $U \subseteq \mathbb{R}^n$ set,

V_u smooth v.f. on M depending cont. $u \in U$.

$\tilde{u} : [0,T] \to U$ is an adm. control if it meas. & essentially bounded. $(\tilde{u} \in L^\infty([0,T], U))$.

For each $\tilde{u}$, let $q_{\tilde{u},q_0}$ be the unique sol.

$$\dot{q}_{\tilde{u},q_0}(t) = V_{\tilde{u}(t)}(q_{\tilde{u},q_0}(t)) \qquad \forall \text{ a.e. } t$$
$$q_{\tilde{u},q_0}(0) = q_0 .$$

We define the attainable sets of q_0 at time $t \geq 0$ as $\mathcal{A}_{q_0}(t) := \{ q_{\tilde{u},q_0}(t) \mid \tilde{u} \text{ adm. control} \}$

$$\mathcal{A}_{q_0}^{t} := \bigcup_{0 \leq \tau \leq t} \mathcal{A}_{q_0}(\tau) .$$

We define Ham. H_u on T^*M for $u \in U$

by $H_u(x,p) := \langle p, V_u(x) \rangle = p(V_u(x))$.

Set $H(\lambda) := \max\limits_{u \in U} H_u(\lambda)$.

Thm (Geom. PMP)

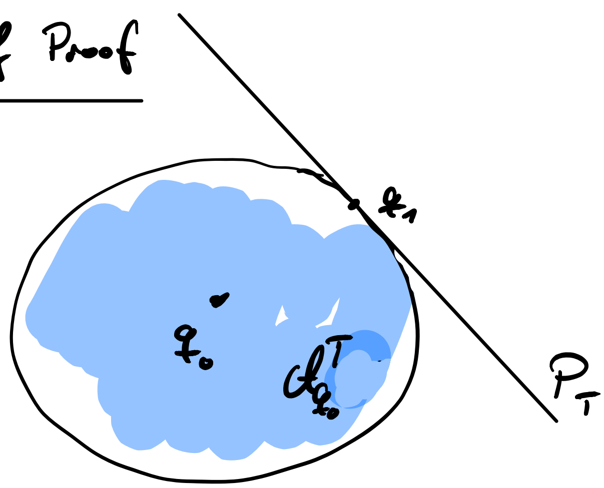
Let $\tilde{u}$ adm. cont. s.th. $q_1 := q_{\tilde{u},q_0}(T) \in \partial \mathcal{A}_{q_0}(T)$.

Then $\exists\ \lambda : [0,T] \to T^*M$ with $\pi \circ \lambda = q_{\tilde{u},q_0}$

and s.th.

$$\left.\begin{array}{ll} (1) & \lambda(t) \neq 0 \\ (2) & \dot{\lambda}(t) = X_{H_{\tilde{u}(t)}}(\lambda(t)) \\ (3) & H_{\tilde{u}(t)}(\lambda(t)) = H(\lambda(t)) \end{array}\right\} \forall \text{ a.e. } t.$$

Sketch of Proof



Assume $\mathcal{A}^T_{q_0}$ convex w/ smooth bdry.

Choose $\lambda_T \in T^*_{q_1} M$ s.th. $P_T = \ker \lambda_T$

For $0 \le t \le T$, observe that $q_{\tilde{u}, q_0}(T) \in \partial \mathcal{A}_{q_0}(T)$ implies that $q_{\tilde{u}, q_0}(t) \in \partial \mathcal{A}_{q_0}(t)$.

$\rightsquigarrow P_t$ analogously, λ_t. To check: We may choose λ_t s.th. λ satisfies (1)–(3).

Can define $\lambda_t := \varphi^*_{t,T}(\lambda_T)$, φ is the flow of time-dep. v.f. $V_{\tilde{u}}$. □

A curve λ as in the Thm. is called Pontryagin extremal.

Prop: Assume H is at least C^2. Then a curve λ in T^*M is a Pontryagin extremal iff $\dot{\lambda} = X_H(\lambda)$.

Pf: „$\Rightarrow$"

Let λ be a Pontr. extr. with control $\tilde{u}$.

By (3), have $H(\lambda(t)) - H_{\tilde{u}(t)}(\lambda(t)) = 0 \quad \forall t$.

By def. of H, we have $H(\mu) - H_{\tilde{u}(t)}(\mu) \ge 0 \quad \forall t, \mu \in T^*M$.

$\Rightarrow d_{\lambda(t)}(H - H_{\tilde{u}(t)}) = 0 \quad \forall t$.

$\Rightarrow X_H(\lambda(t)) = X_{H_{\tilde{u}(t)}}(\lambda(t)) \overset{(2)}{=} \dot{\lambda}(t)$. □

② PMP for Optimal Control

Consider

$$\underset{\substack{\tilde{u} \text{ adm.}\\ \text{control}}}{\text{opt}} \; J(\tilde{u}) = \int_0^T \varphi(q_{\tilde{u},q_0}(t), \tilde{u}(t))\, dt$$

$$q_{\tilde{u},q_0}(T) = q_1$$

opt $\in \{\min, \max\}$

where $\varphi : M \times \mathbb{R}^n \to \mathbb{R}$ is some cost fct.

We form an extended control problem

$$\dot{\hat{q}} = \hat{V}_u(\hat{q}), \quad \hat{q} \in \hat{M}, \quad u \in U$$

$$\hat{q}(0) = \hat{q}_0$$

where $\hat{M} = \mathbb{R} \times M$, $\hat{V}_u(\hat{q}) = (\varphi(q,u), V_u(q))$,

$\hat{q}_0 = (0, q_0)$.

If $q_{\tilde{u},q_0}$ is the sol. for the initial problem, then

$$\hat{q}_{\tilde{u},\hat{q}_0}(t) := \left(J_{\tilde{u}}(t), q_{\tilde{u},q_0}(t)\right) \text{ solves the new problem.}$$

$$J_{\tilde{u}}(t) := \int_0^t \varphi(q_{\tilde{u},q_0}(\tau), u(\tau))\, d\tau. \qquad (J(\tilde{u}) = J_{\tilde{u}}(T)).$$

<u>Prop</u>: Let $\tilde{u}$ is a sol. of the OCP, then

$$\hat{q}_{\tilde{u},q_0}(T) \in \partial \mathcal{A}_{\hat{q}_0}(T).$$

We want to only reach $\partial \mathcal{A}$ if $\tilde u$ is a min.

$\rightsquigarrow$ New problem:

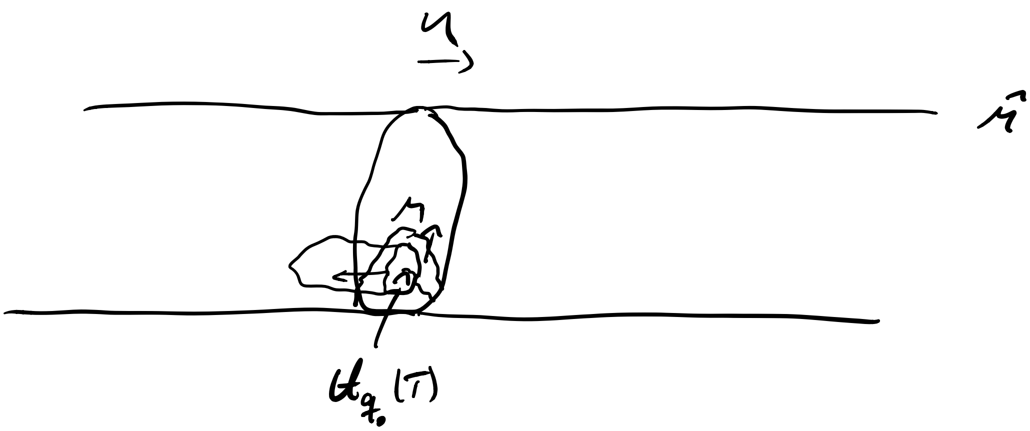
$\dot{\bar q} = \bar V_{u,v}(\bar q)$, $\bar q \in \bar M = \hat M = \mathbb{R} \times M$, $u \in \mathcal{U}$, $v \geq 0$

$\bar q = \bar q_0$, $\bar q_0 = (0, q_0)$

$\bar V_{u,v}(\bar q) = \big(\varphi(q,u) + v \ , V_u(q)\big)$

If $\tilde u$ is adm. cont. for the original prob., then $(\tilde u, 0)$ is adm. cont. for the new probl.

$\bar q_{(\tilde u, 0), \bar q_0}(T) \in \partial \mathcal{A}_{\bar q_0}(T)$ if $\tilde u$ is a min. for the OCP.



Applying PMP to (T) we get a curve μ in $T^*\bar M$ with $\pi \circ \mu = \bar q_{(\tilde u, 0), \bar q_0}$ and

$\dot\mu = X_{H_{\tilde u, \tilde v}}(\mu)$ (2) , $\mu \neq 0$ (1) , $H_{\tilde u, \tilde v}(\mu) = \max_{(u,v)} H_{u,v}(\mu)$ (3).

Write $\mu = (\eta, \lambda) \in T^*_{(y,q)} \bar{M} = \mathbb{R} \oplus T^*_q M$.

$$H_{u,v}(\mu) = H_u(\lambda) + \eta\,(\varphi(q,u) + v).$$

(3) implies $\eta \leq 0$

(2) $\to \dot{\eta} = 0$

(2) $\Rightarrow \dot{\lambda} = X_{H_u^\eta}(\lambda)$, where $H_u^\eta(\lambda) = H_u(\lambda) + \eta \cdot \varphi(q,u)$

__Thm__ (PMP for OCP):

Let $\tilde{u}$ be a min. for OCP. Then $\exists\, \eta \in \mathbb{R}_{\leq 0}$ s.th.

$$\left.\begin{aligned} &(\eta, \lambda(t)) \neq 0 \\ &\dot{\lambda}(t) = X_{H^\eta_{\tilde{u}(t)}}(\lambda(t)) \\ &H^\eta_{u(t)}(\lambda(t)) = \max_{u \in U} H^\eta_u(\lambda(t)). \end{aligned}\right\} \forall \text{ a.e. } t$$

If $\eta = 0$, then μ or $\tilde{u}$ is called abnormal. If $\eta < 0$ it is called normal.

__Cor__: Thm that Anna-Maria showed last time.

__Summary__:

- Geom. PMP gives necessary cond. for realizing Opt.

- For OCP, we have auxiliary prob. (w/ horizontal direction) s.th. optimality implies vertical cond.
- If $H = \max_{u \in U} H_u$ is smooth enough, then we can find candidate sol. for OCP by solving $\dot{\lambda} = X_H(\lambda)$.